Necessary and sufficient conditions for the existence of an image with a given code

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The article introduces an image encoding function which is invariant with respect to affine transform. The properties of the encoding funciton are investigated. Necessary and sufficient conditions are found for a given set of numbers to be a code of nonsingulari image.

Keywords: image code, image encoding, affine equivalence.

Introduction

In pattern recognition in is often required to represent an image consisting of a finite number of points in a form of some code. A possible solution is to simply store coordinates of all points. A drawback here is non-invariance of the code under natural transformations, such as translation, rotation, stretching, etc., and binding of the code to a concrete coordinate system. It is reasonable to consider images mapped on each other by such transformations to be equivalent.

V. N. Kozlov introduced an encoding that is invariant with respect to affine transformations ([4, 5]). It was shown that equality of the codes of two planar images is necessary and sufficient for affine equivalence. In our paper we modify Kozlov's encoding and investigate the properties of the new code. In particular we obtain a criterion that describes all tuples that are codes of some planar images.

Necessary and sufficient conditions for for the existence of a three-dimensional image with two given planar projections were proven in [2].

In [5] V. N. Kozlov proposed the encoding ρ that is invariant with respect to planar invertible affine transformations. Assume that an image consists of n points $a_1, \ldots, a_n$. For every 6-tuple $\{i, j, k, l, m, p\}$ the code equals $\rho_{ijk,lmp} = \frac{S(\triangle a_i a_j a_k)}{S(\triangle a_l a_m a_p)}$, where $S(abc)$ is the area of the triangle abc; 6-tuples are somehow ordered and the corresponding values of ρ form the code. Thus the code of the image $\{a_1, \ldots, a_n\}$ consists of $(C_n^3)^2$ components[2] and is obviously redundant. In our paper we study the degree of this redundancy. We introduce a modified encoding and obtain explicit conditions that a tuple of

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[2] A permutation of vertices does not affect triangle area. So the tuples with permuted (i, j, k) and (k, l, m) are considered as the same tuple.

numbers is an encoding of some image. We extend the result to the case of the original encoding (in an implicit form).

The modified encoding on a 6-tuple $\{i,j,k,k,m,n\}$ equals $r_{ijk,lmp} = \frac{S'(\triangle a_i a_j a_k)}{S'(\triangle a_l a_m a_p)}$, where S' is the oriented area, i.e. the area endowed with the sign that corresponds to the direction of vertex traversal.

The rest of the paper is organized as follows. The basic concepts and notation are introduced in Section 1. The properties of the image code matrix are investigated in Section **??**. The main result is formulated and proved in Section **??**. Section **??** is a conclusion.

1. Basic notation

Let S' be the oriented triangle area, i.e. $S'(\triangle abc) = S(\triangle abc)$ for the positive triangle orientation and $S'(\triangle abc) = -S(\triangle abc)$ for the negative one. The triangle orientation is considered to be positive if the triangle vertices are traversed in counterclockwise order and negative otherwise.

Suppose that $n \in \mathbb{N}$ and $a_1, \ldots, a_n$ are some points on a plane. The set $A = \{a_1, \ldots, a_n\}$ is said to be an *image*. An image is *degenerate* if all points are collinear; otherwise it is *non-degenerate*. Fix some (Euclidean) coordinate system. The coodrinates of a point a_i are denoted by $X(a_i)$ and $Y(a_i)$. Hereinafter for the sake of convenience individual indices are denoted by lowercase Latin letters.

Multi-indices, i.e. 3-tuples, are denoted by lowercase Greek letters $\alpha, \beta, \gamma, \ldots$ The components of a multi-index are denoted by $\alpha = [\alpha(1), \alpha(2), \alpha(3)]$, and the triangle[1] with the corresponding vertices is denoted by $\triangle_\alpha = \triangle a_{\alpha(1)} a_{\alpha(2)} a_{\alpha(3)}$.

Let us introduce an equivalence relation on the set of multi-indices. Two multi-indices are equivalent (and thus indistinguishable) if one of them can be obtained from the other by a cyclic permutation. More formally, $\alpha \simeq \alpha'$ if and only if the permutation $\begin{pmatrix} \alpha(1) & \alpha(2) & \alpha(3) \\ \alpha'(1) & \alpha'(2) & \alpha'(3) \end{pmatrix} \in S_3$ is even. A multi-index $\bar{\alpha}$ is conjugate to α if the permutation $\begin{pmatrix} \alpha(1) & \alpha(2) & \alpha(3) \\ \bar{\alpha}(1) & \bar{\alpha}(2) & \bar{\alpha}(3) \end{pmatrix}$ is odd. In essence equivalence means that the corresponding triangles have equal areas, whereas conjugation means that the areas of the corresponding triangles differ only in sign. Hereinafter for the sake of brevity the term "multi-index" will be used to identify equivalence classes.

In total, there are C_n^3 different unoriented triangles with vertices from $A = \{a_1, \ldots, a_n\}$, and respectively $N = 2 \cdot C_n^3$ oriented ones, so there are $2C_n^3$ multi-index equivalence classes.

[1] This notation is also used for the case of degenerate triangles, i.e. collinear points.

Enumerate all multi-indices (i.e. the corresponding triangles $\alpha_1, \ldots, \alpha_N$). Let $\mathcal{A} = \{\alpha_1, \ldots, \alpha_N\}$ be the set of all multi-indices and $E : \alpha_i \mapsto i$ be the enumeration function.

Consider the set of all fractions $r_{ijk,lmp} = \frac{S'(\triangle a_i a_j a_k)}{S'(\triangle a_l a_m a_p)}$. If a triangle $\triangle a_l a_m a_p$ is degenerate, i.e. $S'(\triangle a_l a_m a_p) = 0$, then $r_{ijk,lmp} = \infty$. This set is referred to as the *code* of the image $\{a_1, \ldots, a_n\}$. A similar encoding procedure was proposed in [1].

Definition 1. Let $R = (r_{ij})$ be the matrix of the size $N \times N$ such that the element at the intersection of the ith row and jth column is $r_{ij} = r_{\alpha_i,\alpha_j} = \frac{S'(\triangle_\alpha)}{S'(\triangle_\beta)}$. Thus the elements of the code are arranged into a square table with rows and columns enumerated by multi-indices (triangles). The matrix R is the *code matrix* of the image.

Remark 1. Hereinafter we will also use the notation $r_{\alpha\beta} = R_{E(\alpha)E(\beta)}$, i.e. enumerate rows and columns of code matrices by multi-indices.

Example 1. Consider a trapezoid $a_1a_2a_3a_4$ with bases a_1a_2 and a_4a_3 such that $|a_1a_2| : |a_4a_3| = 1 : 2$ (see Fig. 1). Enumerate multi-indices as it is

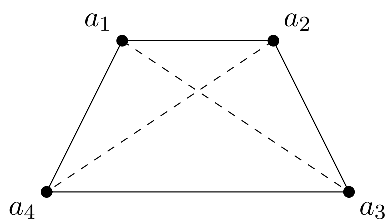

Fig. 1. Example 1.

shown in Table 1. Note that the last four multi-indices are conjugate to the first four, so it is sufficient to construct only the part of the image code matrix corresponding to the first 4 rows and columns. This submatrix is the following: $R_4 = \begin{pmatrix} 1 & 1 & 1/2 & 1/2 \\ 1 & 1 & 1/2 & 1/2 \\ 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \end{pmatrix}$. The complete code matrix has the following form: $R = \begin{pmatrix} R_4 & -R_4 \\ -R_4 & R_4 \end{pmatrix}$.

Remark 2. Let us enumerate multi-indices in such an order that the second half of the multi-index list is an element-wise conjugation of the first half. More formally $\alpha_{i+N/2} = \overline{\alpha_i}$ for $i = 1, ..., N/2$. Then the code matrix will have the similar form $R = \begin{pmatrix} R' & -R' \\ -R' & R' \end{pmatrix}$.

i	α_i
1	$1, 2, 3$
2	$1, 2, 4$
3	$1, 3, 4$
4	$2, 3, 4$
5	$1, 3, 2$
6	$1, 4, 2$
7	$1, 4, 3$
8	$2, 4, 3$

Table 1. Multi-index enumeration table

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